Derivation of a compressible bubbly flow model

CJC - MA
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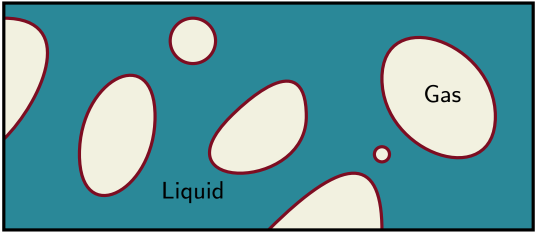
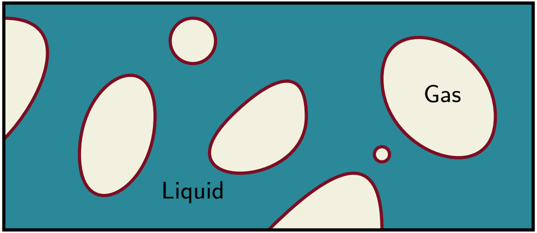
Motivations



Context

- Nuclear safety: accident situation in pressurized water reactor
- High temperature and pressure conditions
- Compressible two-phase flows (or multiphase flows) with heterogeneities (bubbles, droplets...)
- Thermodynamical (dis)equilibrium
  - Thermal, mechanical, mass transfers through the interface
- Complex wave interactions (shocks, phase transition...)

Motivations



- Macroscopic description
  - Derive average models
- Implicit description of the interface
- Compressible behaviour
- Mathematical structure
  - Well-posedness (smooth solutions, discontinuities...)

The richest model [Baer, Nunziato '86]

- Euler system for two immiscible phases $k = f, g$ with coupling terms
- Phasic state $w_k = (\rho_k, u_k)$, $w = (w_f, w_g)$
- Pressure $p_k = p_k(\rho_k)$, chemical potential $g_k = g_k(\rho_k)$
- Volume fraction $\alpha_k \in [0, 1]$

$$
\begin{aligned}
&\alpha_f + \alpha_g = 1 \qquad \rho = \alpha_f \rho_f + \alpha_g \rho_g \\
&\partial_t \alpha_f + u_i(w) \partial_x \alpha_f = \lambda_p(w)(p_f - p_g) \\
&\partial_t (\alpha_k \rho_k) + \partial_x (\alpha_k \rho_k u_k) = \lambda_\rho(w)(g_l - g_k) \\
&\partial_t (\alpha_k \rho_k u_k) + \partial_x (\alpha_k \rho_k u_k^2 + \alpha_k p_k) - p_i(w) \partial_x \alpha_k = \lambda_u(w)(u_l - u_k)
\end{aligned}
$$

✓ Well-posedness although nonconservative: hyperbolicity, symmetrizable, jump conditions...

✗ Heuristic closure laws for interfacial quantities $u_i(w)$ and $p_i(w)$

✗ Empirical source term (and relaxation time scales $\lambda_{p,\rho,u}(w)$ as well)

Derivation of averaged models

- **Averaging approach**
  [Drew & Passman '98, Ishii & Hibiki '06,...]
  - Microscopic description
    - Instantaneous local conservation laws for each separated phase
    - Jump conditions through the interfaces
  - Averaging process
    - Introduce time and/or volume scales, or random disturbances
    - Average the microscopic model wrt the small scales

  ✓ Baer-Nunziato type model

  ✗ Closure laws
  ✗ Definition of the averaging operators

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  ✗ Closure laws
  ✗ Definition of the averaging operators

- **Homogenization approach**
  [Serre '91 & '01, E '92, Hillairet '07, Bresch & Huang '11, Bresch, Hillairet '15 & '19, Hillairet '18, Bresch, Burtea & Lagoutière '20,...]

Homogenization for two-phase flows

Standard approach

- Microscopic description
  - Viscous flows (smooth enough solutions) for both phase
  - Conditions through the interfaces ("perfect transducers")
- One-fluid model with high-oscillatory density solutions
- Pass to the limit to deduce macroscopic quantities $\bar{\alpha}_{f,g}, \bar{\rho}_{f,g}, \bar{\rho}$ and $\bar{u}$

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$$
\begin{cases}
\bar{\alpha}_f + \bar{\alpha}_g = 1, \qquad \bar{\rho} = \bar{\alpha}_f \bar{\rho}_f + \bar{\alpha}_g \bar{\rho}_g \\
\partial_t \bar{\alpha}_f + \bar{u} \partial_x \bar{\alpha}_f = \dfrac{\bar{\alpha}_g \bar{\alpha}_f}{\bar{\alpha}_f \mu_g + \bar{\alpha}_g \mu_f} \Big[(\mu_g - \mu_f) \partial_x \bar{u} + (\mathrm{p}_f(\bar{\rho}_f) - \mathrm{p}_g(\bar{\rho}_g)) \Big] \\
\partial_t (\bar{\alpha}_f \bar{\rho}_f) + \partial_x (\bar{\alpha}_f \bar{\rho}_f \bar{u}) = 0, \qquad \partial_t (\bar{\alpha}_g \bar{\rho}_g) + \partial_x (\bar{\alpha}_g \bar{\rho}_g \bar{u}) = 0 \\
\partial_t (\bar{\rho} \bar{u}) + \partial_x (\bar{\rho} \bar{u}^2) = \partial_x \bar{\Sigma} \\
\text{with } \bar{\Sigma} = \frac{\mu_g \mu_f}{\bar{\alpha}_f \mu_g + \bar{\alpha}_g \mu_f} \Big[\partial_x \bar{u} - \Big(\frac{\bar{\alpha}_f}{\mu_f} \mathrm{p}_f(\bar{\rho}_f) + \frac{\bar{\alpha}_g}{\mu_g} \mathrm{p}_g(\bar{\rho}_g) \Big) \Big]
\end{cases}
$$

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- Microscopic description
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- Pass to the limit to deduce macroscopic quantities $\bar\alpha_{f,g}, \bar\rho_{f,g}, \bar\rho$ and $\bar u$

$$\begin{cases} \bar\alpha_f + \bar\alpha_g = 1, \qquad \bar\rho = \bar\alpha_f\bar\rho_f + \bar\alpha_g\bar\rho_g \\ \partial_t\bar\alpha_f + \bar u\partial_x\bar\alpha_f = \dfrac{\bar\alpha_g\bar\alpha_f}{\bar\alpha_f\mu_g + \bar\alpha_g\mu_f}\Big[(\mu_g-\mu_f)\partial_x\bar u + (\mathrm{p}_f(\bar\rho_f) - \mathrm{p}_g(\bar\rho_g))\Big] \\ \partial_t(\bar\alpha_f\bar\rho_f) + \partial_x(\bar\alpha_f\bar\rho_f\bar u) = 0, \qquad \partial_t(\bar\alpha_g\bar\rho_g) + \partial_x(\bar\alpha_g\bar\rho_g\bar u) = 0 \\ \partial_t(\bar\rho\bar u) + \partial_x(\bar\rho\bar u^2) = \partial_x\bar\Sigma \\ \text{with } \bar\Sigma = \frac{\mu_g\mu_f}{\bar\alpha_f\mu_g+\bar\alpha_g\mu_f}\left[\partial_x\bar u - \left(\frac{\bar\alpha_f}{\mu_f}\mathrm{p}_f(\bar\rho_f) + \frac{\bar\alpha_g}{\mu_g}\mathrm{p}_g(\bar\rho_g)\right)\right] \end{cases}$$

Pros & Cons

✓ Fully rigorous

✗ Simple interface behaviors, one-velocity models

- Extensions: different EoS [Bresch & Hillairet '19], temperature [Hillairet '21], density overlap [Bresch, Burtea & Lagoutière '20]

Homogenization for two-phase flows

Standard approach

- Microscopic description
  - Viscous flows (smooth enough solutions) for both phase
  - Conditions through the interfaces ("perfect transducers") ✗
- One-fluid model with high-oscillatory density solutions ✗
- Pass to the limit to deduce macroscopic quantities $\bar{\alpha}_{f,g}, \bar{\rho}_{f,g}, \bar{\rho}$ and $\bar{u}$

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\begin{cases}
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\partial_t \bar{\alpha}_f + \bar{u}\partial_x \bar{\alpha}_f = \dfrac{\bar{\alpha}_g\bar{\alpha}_f}{\bar{\alpha}_f\mu_g + \bar{\alpha}_g\mu_f}\Big[(\mu_g - \mu_f)\partial_x \bar{u} + (\mathrm{p}_f(\bar{\rho}_f) - \mathrm{p}_g(\bar{\rho}_g)) + \ldots\Big] \\
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\partial_t(\bar{\rho}\bar{u}) + \partial_x(\bar{\rho}\bar{u}^2) = \partial_x\bar{\Sigma} \\
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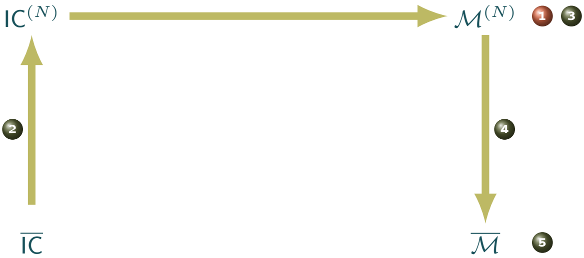
+\text{Additional relations}
\end{cases}
$$

Goal [Hillairet, M. & Seguin '23]

- Introduce more complex interface behavior
- Couple different fluid models

Outline/Methodology



1. Microscopic model with N bubbles
2. Macroscopic to microscopic initial data
3. Solve the microscopic model
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The one-dimensional microscopic model

Assumptions for the microscopic model (first in 3D)

- Fluid f, N bubbles of gas g, $k \in \{1, \dots, N\}$
- Compressible Navier-Stokes equations for both phases
- Bubbles remain spherical (translation, dilatation, rotation)
- Interface conditions
  - Continuity of the velocity field $u_f = u_k$
  - Surface tension $(\Sigma_f - \Sigma_n)\nu = \kappa_n \nu$

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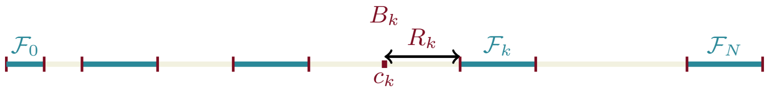
- Interface conditions
  - Continuity of the velocity field $\quad u_f = u_k$
  - Surface tension $\quad (\Sigma_f - \Sigma_n) = \gamma / R_k$

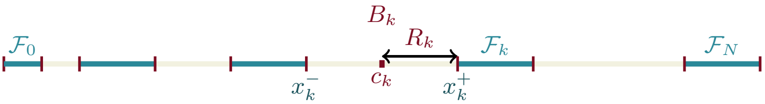
In 1D



- In the fluid domain $\mathcal{F}(t)$
  - ρ_f, u_f, $\Sigma_f = \mu_f \partial_x u_f - \mathrm{p}_f(\rho_f)$
- In each buble $B_k(t) = B(c_k(t), R_k(t))$

The one-dimensional microscopic model

In 1D



In the fluid domain $\mathcal{F}(t)$

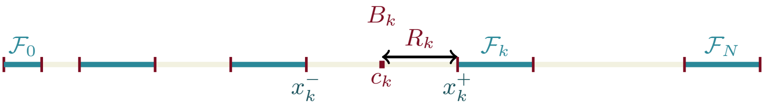
$$\begin{cases} \partial_t \rho_f + \partial_x(\rho_f u_f) = 0 \\ \partial_t(\rho_f u_f) + \partial_x(\rho_f (u_f)^2) = \partial_x \Sigma_f \\ \text{with } \Sigma_f = \mu_f \partial_x u_f - \mathrm{p}_f(\rho_f) \end{cases}$$

In each buble $B_k(t) = B(c_k(t), R_k(t))$ of (constant) mass m_k:

$$\begin{cases} m_k \ddot{c}_k = \Sigma_f(t, x_k^+) - \Sigma_f(t, x_k^-) \\ \dfrac{m_k}{3} \ddot{R}_k = \Sigma_f(t, x_k^-) + \Sigma_f(t, x_k^+) - 2\Sigma_k + \dfrac{\gamma_s}{R_k} \\ \text{with } \Sigma_k = \mu_g \partial_x u_k - \mathrm{p}_g\left(\dfrac{m_k}{2R_k}\right) \end{cases}$$

The one-dimensional microscopic model

In 1D



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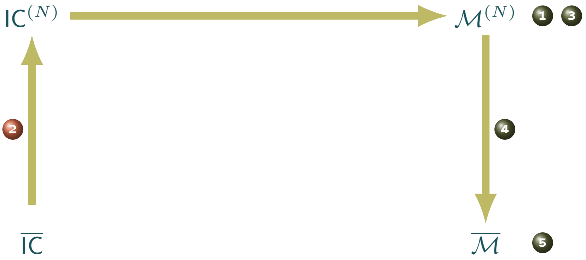
- Well posedness of the Cauchy problem for a time $T > 0$, depending on N...

Outline/Methodology



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Macroscopic to microscopic initial data

At the macroscopic scale, both fluids are present everywhere in the domain Ω

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Macroscopic initial data

- Density of the fluid $\bar{\rho}_f^0 \geq \rho_{\min}$
- Density of the gas $\bar{\rho}_g^0 \geq \rho_{\min}$
- Mean velocity $\bar{u}^0$
- Void fraction $\bar{\alpha}_g^0$

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Reconstruction of gas-bubble distribution

- Probability distribution of the bubbles, in position x and radius r

$$\bar{S}_g^0 = \bar{S}_g^0(x, r) \in L^1(\Omega \times \mathbb{R}^+)$$

Moments of the probability distribution $\bar{S}_g^0$:

- $1-$st order moment $\rightsquigarrow$ void fraction

$$\bar{\alpha}_g^0(x) = \int_{\mathbb{R}+} (2r)\bar{S}_g^0(x, r)\mathrm{d}r$$

- $0-$th order moment ($\rightsquigarrow$ gas covolume or "interfacial area")

$$\bar{f}_g^0(x) = \int_{\mathbb{R}+} \bar{S}_g^0(x, r)\mathrm{d}r$$

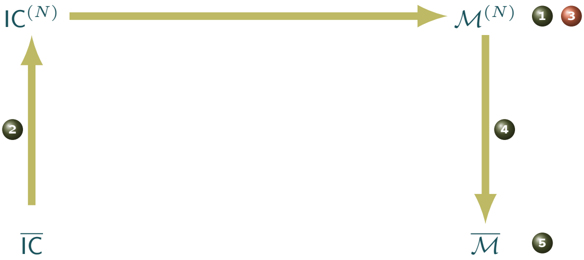
Macroscopic to microscopic initial data

Family of microscopic initial data to be constructed from $\bar{S}_g^0$, $\bar{\rho}_{f,g}^0$ and $\bar{u}^0$

For any bubble number $N \geq 1$:

1. Define a bubble distribution from $\bar{S}_g^0$ to get $(c_k^{(N)}, R_k^{(N)})_{k=1,\dots,N}$
2. Define the densities
$$\begin{cases} \rho_f^{(N)}(0,x) & \text{on } \mathcal{F}^{(N)}(0) \\ \rho_k^{(N)}(0,x) & \text{on } B_k^{(N)}(0) \end{cases}$$
3. Define the velocities
$$\begin{cases} u_f^{(N)}(0,x) & \text{on } \mathcal{F}^{(N)}(0) \\ \dot{c}_k^{(N),0} \text{ and } \dot{R}_k^{(N),0} & \text{on } B_k^{(N)}(0) \end{cases}$$

Outline/Methodology



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Solve the microscopic model

Scaling

- m_k, R_k^0, $|\mathcal{F}_k^0|$ and γ_s behave as N^{-1}

Microscopic Cauchy problem, independent of N [Hillairet, M., Seguin '23]

Consider compatible initial data and the scaling. Then there exists $T_\infty > 0$, independent of N, such that

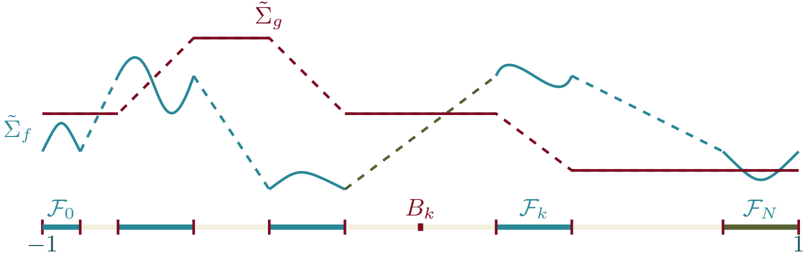
$$\left((c_k^{(N)}, R_k^{(N)})_{k=1,\dots,N}, \rho_f^{(N)}, u_f^{(N)}, (\rho_k^{(N)}, u_k^{(N)})_{k=1,\dots,N}\right)$$

exists and is unique

- T_∞ taken smaller and smaller along the proof
- Combine energy and regularity estimates, independent of N
- Smoothness of the velocity $u_f^{(N)}$ obtained by extended stress tensors

Extended stress tensors

Linear extension of Σ_f and Σ_g over the whole domain Ω



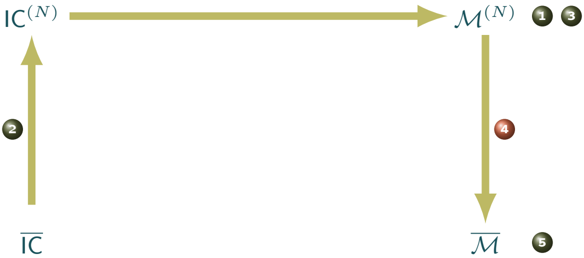
$\rightsquigarrow$ Distinct stress tensors

Tensor estimates

$$\int_0^t \left[\|\tilde{\Sigma}_f\|^2_{H^1(\Omega)} + \|\tilde{\Sigma}_g\|^2_{H^1(\Omega)} + \sum_{k=1}^{N} m_k \left(|\ddot{R}_k|^2 + |\ddot{c}_k|^2 \right) \right] \mathrm{d}s \leq K$$

with K independant of N (for $t < T_\infty$ sufficiently small)

Outline/Methodology



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Pass to the limit $N \to \infty$ to deduce macroscopic quantities

Fluid unknowns

- Characteristic function of the fluid domain

$$\chi^{(N)} = \mathbf{1}_{\mathcal{F}^{(N)}} \rightharpoonup \bar{\alpha}_f L^{\infty}((0,T) \times \Omega)) - w*$$

- Density $\rho_f^{(N)} \longrightarrow \bar{\rho}_f \in L^2((0,T) \times \Omega)$

Gas unknowns

- Density $\rho_g^{(N)} = \sum_{k=1}^{N} \rho_k^{(N)} \mathbf{1}_{B_k} \longrightarrow \bar{\rho}_g$
- Interfacial area/covolume $f^{(N)} = \sum_{k=1}^{N} \left(\frac{1}{2NR_k} \right) \mathbf{1}_{B_k}$

$$f^{(N)} \longrightarrow \bar{f}_g \in L^2((0,T) \times \Omega)$$

Mixture unknowns

- Density $\rho^{(N)} \longrightarrow \bar{\rho}$ and velocity $\tilde{u}^{(N)} \longrightarrow \bar{u}$

Pass to the limit $N \to \infty$ to deduce macroscopic quantities

Fluid unknowns

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Mixture unknowns

- Density $\rho^{(N)} \longrightarrow \bar{\rho}$ and velocity $\tilde{u}^{(N)} \longrightarrow \bar{u}$

Methodology

- Use various estimates to prove relative compactness (up to extraction of subsequences)
- Each unknown satisfy an evolution equation

Outline/Methodology

1. Microscopic model with N bubbles
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Macroscopic evolution equations

Which evolution equations do we expect?

- Immiscibility constraint
- Void fraction equation $\bar{\alpha}_{f,g}$, accounting for mechanical relaxation
- Partial mass conservations with $\bar{\alpha}_f \bar{\rho}_f$ and $\bar{\alpha}_g \bar{\rho}_g$
- Momentum equation with $\bar{\rho}\bar{u}$ with mixture density $\bar{\rho}$

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- Momentum equation with $\bar{\rho}\bar{u}$ with mixture density $\bar{\rho}$

Methodology

⇝ Pass to the limit in nonlinear combinations of $\chi^{(N)}(t,x)$, $\rho^{(N)}(t,x)$ and $f_g^{(N)}(t,x)$

✓ Nonlinear convergence (in the sense of Young measures)

The key ingredient

Nonlinear function $b \in C^1([0,1] \times \mathbb{R}^+ \times \mathbb{R}^+)$ and consider the sequence

$$b^{(N)}(t,x) = b(\chi^{(N)}(t,x), \rho^{(N)}(t,x), f_g^{(N)}(t,x)) \quad \forall (t,x) \in (0,T) \times \Omega$$

Nonlinear convergence

The sequence $(b^{(N)})$ satisfies

$$\partial_t b^{(N)} + \partial_x (b^{(N)} \tilde{u}^{(N)}) + \big(\partial_2 b^{(N)} \rho^{(N)} + \partial_3 b^{(N)} f_g^{(N)} - b^{(N)}\big) \partial_x \tilde{u}^{(N)} = 0 \quad \text{in } \mathcal{D}'((0,T) \times \Omega)$$

Moreover, there exists $\bar{b} \in L^\infty((0,T) \times \Omega)$ such that

$$b^{(N)} \rightharpoonup \bar{b}, \quad \text{in } L^\infty((0,T) \times \Omega) - w^\star \text{ when } N \to +\infty$$

Some examples

Considering $b^{(N)} = \chi^{(N)}$ and $b^{(N)} = 1 - \chi^{(N)}$ gives $\bar{b} = \bar{\alpha}_f$ and $\bar{b} = \bar{\alpha}_g$

The immiscibility constraint holds

$$\bar{\alpha}_f + \bar{\alpha}_g = 1$$

Some examples

Considering $b^{(N)} = \chi^{(N)}$ and $b^{(N)} = 1 - \chi^{(N)}$ gives $\bar{b} = \bar{\alpha}_f$ and $\bar{b} = \bar{\alpha}_g$

The immiscibility constraint holds

$$\bar{\alpha}_f + \bar{\alpha}_g = 1$$

Considering $b^{(N)} = f_g^{(N)}$ gives $\bar{b} = \bar{f}_g$

The interfacial area $\bar{f}_g$ satisfies

$$\partial_t \bar{f}_g + \partial_x (\bar{f}_g \bar{u}) = 0$$

The macroscopic model

Macroscopic closed system

$$
\begin{cases}
\bar{\alpha}_f + \bar{\alpha}_g = 1, \qquad \bar{\rho} = \bar{\alpha}_f \bar{\rho}_f + \bar{\alpha}_g \bar{\rho}_g \\
\partial_t \bar{\alpha}_f + \bar{u} \partial_x \bar{\alpha}_f = \dfrac{\bar{\alpha}_g \bar{\alpha}_f}{\bar{\alpha}_f \mu_g + \bar{\alpha}_g \mu_f} \left[(\mu_g - \mu_f) \partial_x \bar{u} + (\mathrm{p}_f(\bar{\rho}_f) - \mathrm{p}_g(\bar{\rho}_g)) - \bar{\gamma}_s \dfrac{\bar{f}_g}{\bar{\alpha}_g} \right] \\
\partial_t \bar{f}_g + \partial_x (\bar{f}_g \bar{u}) = 0 \\
\partial_t (\bar{\alpha}_f \bar{\rho}_f) + \partial_x (\bar{\alpha}_f \bar{\rho}_f \bar{u}) = 0, \qquad \partial_t (\bar{\alpha}_g \bar{\rho}_g) + \partial_x (\bar{\alpha}_g \bar{\rho}_g \bar{u}) = 0 \\
\partial_t (\bar{\rho} \bar{u}) + \partial_t (\bar{\rho}^2 \bar{u}) = \partial_x \bar{\Sigma} \\
\text{with } \bar{\Sigma} = \frac{\mu_g \mu_f}{\bar{\alpha}_f \mu_g + \bar{\alpha}_g \mu_f} \left[\partial_x \bar{u} - \left(\frac{\bar{\alpha}_f}{\mu_f} \mathrm{p}_f(\bar{\rho}_f) + \frac{\bar{\alpha}_g}{\mu_g} \mathrm{p}_g(\bar{\rho}_g) \right) - \frac{\bar{\gamma}_s}{\mu_g} \bar{f}_g \right]
\end{cases}
$$

The macroscopic model

Macroscopic closed system

$$\begin{cases} \bar{\alpha}_f + \bar{\alpha}_g = 1, \qquad \bar{\rho} = \bar{\alpha}_f\bar{\rho}_f + \bar{\alpha}_g\bar{\rho}_g \\ \partial_t\bar{\alpha}_f + \bar{u}\partial_x\bar{\alpha}_f = \dfrac{\bar{\alpha}_g\bar{\alpha}_f}{\bar{\alpha}_f\mu_g + \bar{\alpha}_g\mu_f}\left[(\mu_g - \mu_f)\partial_x\bar{u} + (\mathrm{p}_f(\bar{\rho}_f) - \mathrm{p}_g(\bar{\rho}_g)) - \bar{\gamma}_s\dfrac{\bar{f}_g}{\bar{\alpha}_g}\right] \\ \partial_t\bar{f}_g + \partial_x(\bar{f}_g\bar{u}) = 0 \\ \partial_t(\bar{\alpha}_f\bar{\rho}_f) + \partial_x(\bar{\alpha}_f\bar{\rho}_f\bar{u}) = 0, \qquad \partial_t(\bar{\alpha}_g\bar{\rho}_g) + \partial_x(\bar{\alpha}_g\bar{\rho}_g\bar{u}) = 0 \\ \partial_t(\bar{\rho}\bar{u}) + \partial_t(\bar{\rho}^2\bar{u}) = \partial_x\bar{\Sigma} \\ \text{with } \bar{\Sigma} = \frac{\mu_g\mu_f}{\bar{\alpha}_f\mu_g + \bar{\alpha}_g\mu_f}\left[\partial_x\bar{u} - \left(\frac{\bar{\alpha}_f}{\mu_f}\mathrm{p}_f(\bar{\rho}_f) + \frac{\bar{\alpha}_g}{\mu_g}\mathrm{p}_g(\bar{\rho}_g)\right) - \frac{\bar{\gamma}_s}{\mu_g}\bar{f}_g\right] \end{cases}$$

Additional kinetic equation

- Distribution function in position and radius $S_t^{(N)} = \dfrac{1}{N}\sum_{k=1}^{N}\delta_{c_k(t),NR_k(t)}$

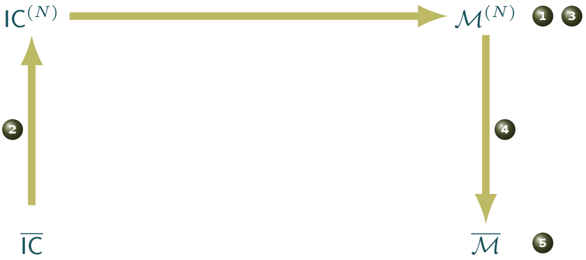
$$\langle S_t^{(N)}, \beta\rangle \to \langle \bar{S}_g, \beta\rangle, \quad \text{in } C([0,T])$$

- Probability distribution $\bar{S}_g$ satisfies

$$\partial_t\bar{S}_g - \partial_x(\bar{S}_g\bar{u}) + \frac{1}{\mu_g}\partial_r\big((r(\bar{\Sigma}_g + \mathrm{p}_g(\bar{\rho}_g)) + \bar{\gamma}_s/2\big)\bar{S}_g) = 0$$

- 0-th order moment $\bar{f}_g(x) = \int_{\mathbb{R}+}\bar{S}_g(x,r)\mathrm{d}r$

Conclusion



1. Microscopic model with N bubbles
2. Macroscopic to microscopic initial data
3. Solve the microscopic model
4. Pass to the limit $N \to \infty$ to deduce macroscopic quantities
5. Find the associated macroscopic evolution equations

To sum up

Comments on the macroscopic model

- Two-pressure one-velocity two-phase flow model
  - Both phases are compressible and viscous
  - Extension of Bresch & Hillairet models: mechanical relaxation, surface tension, not a “one-fluid" model
- Additional description
  - New variable $\bar{f}_g$: interfacial area in 3D?
  - Kinetic equation on the probability distribution $\bar{S}_g$ wrt (t, x, r)

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- Comparison with other bubbly flow models
- 3D extension, at least formal...
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Thank you for your attention!